ESC103 Unit 21

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1 Finding the inverse using Gaussian elimination

In Unit 12, we introduced elimination matrices and showed how the elimination algorithm presented in unit 11 for solving square systems $A\overrightarrow{X} = \overrightarrow{b}$ can also be done using elimination matrices combined with matrix multiplication (on the left)

Allowable elementary operations:

I) Interchange rows

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

What would we have to multiply by in order to perform this row interchange?:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

We can ind that elimination matrix by performing the same row operation on the identity matrix.

II) Multiply one row by a constant:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 8c & 8d \end{bmatrix}$$

Can we convert this to a multiplication of matricies?:

$$\begin{bmatrix} 1 & 0 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 8c & 8d \end{bmatrix}$$

We found that elimination matrix by applying the allowable elementary operation on the identity matrix.

II) Add or subtract a multiple of one row to or from another row.

> The matrices associated with these 3 permitted elementary operations (**I, II, III**), are called **elementary matrices**.

2 Algorithm for finding A^{-1} (if it exists)

We don't know if A^{-1} exists even, but we will proceed and see what happens.

Start with the following "super" augmented matrix:

$$[A|I]$$

Start applying G.E. to bring matrix A to its RREF,

$$\rightarrow [E_1, A|E_1, I]$$

$$\rightarrow [E_2 E_1, A|E_2 E_1, I]$$

$$...$$

$$...$$

let's say it took k steps to get to RREF.

$$\rightarrow [E_k, E_{k-1}..., A|E_k, E_{k-1}..., I]$$

If after k elementary operations:

$$E_kE_{k-1}...E_2E_1, A = I_{\text{(identity matrix)}}$$

Is the RREF the identity matrix?

Think of $E_kE_{k-1}...E_2E_1$ as one matrix, we can then say that this product $= A^{-1}$

$$[A|I] \rightarrow_{\text{GE}} [I|A]$$

We're basically left with the inverse now on the right side of our super augmented matrix.

If RREF is NOT the identity matrix, that simply means that matrix A isn't invertible.

So not only have we used GE to find A^{-1}, we have also used it to express A^{-1} as a product of elementary matrices.

What about matrix A?

Every elementary matrix (E) is invertible, and E^{-1} can be found by applying the reverse of the elementary operation that produced E, to I.

$$E^{-1}$$

is also an elementary matrix!

because:

$$AA^{-1} = A^{-1}A = I$$

$$\therefore (A^{-1})^{-1} = A = (E_kE_{k-1}...E_2E_1)^{-1}$$

$$= E_1^{-1}E_2^{-1}....E_{k-1}^{-1}E_k^{-1}$$

3 An example and a segue into MAT185

Find the inverse of matrix A using G.E. (assuming the inverse exists).

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[\ A \mid I\ ]$$

$$\left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \text{R3 + 3R4}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \text{R2 + 2R3}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \text{R1 + R2}$$

$$\rightarrow \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1 & 2 & 6 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = I$$

$$A^{-1} = \begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & 1 & 2 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remember, if you don't get the identity matrix on the left side of the super augment matrix, then matrix A is not invertible.

$$(A^{-1}A)\overrightarrow{x} = A^{-1}\overrightarrow{b}$$
$$\overrightarrow{x} = A^{-1}\overrightarrow{b}$$

$$A\overrightarrow{x} = \overrightarrow{b}$$

Let's say we want to find the associated elementary matrix, we can do this by applying the elementary operations.

$$E_1 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{R3 + 3R4}$$

$$\rightarrow E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E2 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{R2 + 2R3}$$

$$\rightarrow E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$E3 : \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{R2 + 2R3}$$

$$\rightarrow E_3 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore E_3 E_2 E_1 A = I$$

$$\therefore A^{-1} = E_3 E_2 E_1$$

$$\therefore A = E_1^{-1} E_2^{-1} E_3^{-1}$$

Let's revisit $A = CR$:
From earlier, we know R_0 denotes the RREF of matrix A.

In one example, $R = R_0$ (because R_1 has no zero rows).

Consider the following $A = CR$
A $= 4 \times 4$
C $= 4 \times 4$
R $= 4 \times 4$

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \\ ? & ? & ? & ? \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can see quite easily that C is A, since we want SOMETHING times the identity matrix to equal A, therefore that thing is A.

In closing, we are now in a position to make a number of equivalent statements:

a) Matrix A is invertible
b) $A\overrightarrow{x} = \overrightarrow{b}$ has one and only one solution (unit 20: note 2)
c) $A\overrightarrow{x} = \overrightarrow{0}$ has only the trivial solution (unit 20: note 3)
d) The RREF of matrix A is I (unit 21)
e) Matrix A is expressible as a product of elementary matrices.
f) The column vectors of matrix A are independent (unit 22)

ALL THESE STATEMENTS MUST BE TRUE TOGETHER, IF ONE IS KNOWN TO BE TRUE OR FALSE, THE OTHERS CAN BE KNOWN CERTAINLY.